A CLASS OF PERIODIC ORBITS OF AN INFINITESIMAL BODY SUBJECT TO THE ATTRACTION OF n FINITE BODIES*

BY

WILLIAM RAYMOND LONGLEY

Introduction.

The question of periodic solutions of a set of differential equations with application to the problem of three bodies has been treated by POINCARÉ,† and the idea of analytic continuation which he developed will be employed in this paper to prove the existence of the solutions under consideration.

Suppose that n finite bodies move in one plane and that their coördinates with reference to one of the bodies, M, as origin are known functions of the time. At any epoch t_0 the bodies form a certain geometrical figure and have certain relative components of velocity. Suppose that at the epoch $t_0 + T_1$ the bodies form again the same geometrical figure and have the same relative components of velocity; it is said that the motion is periodic with the period T_1.

A particle P is supposed to move in the plane of the motion of the finite bodies, subject to their newtonian attraction. In the differential equations of motion of the particle it is convenient to divide the terms into two classes: (1) those which are due to the attraction of M alone, and, (2) those which are due to the attraction of the other bodies. A parameter μ is introduced as a factor of all the terms of the second class, so that when $\mu = 0$ the differential equations represent the motion of P when subject to the attraction of only one body, M. In this case the solution is known and the conic in which the particle moves is called the *undisturbed orbit.*

If there exists a periodic solution of the differential equations when $\mu \neq 0$,‡ the period T of the solution must be a multiple of T_1, $T = qT_1$. Suppose the period is assigned in advance. When $\mu = 0$ a periodic solution is known. The initial conditions may be determined so that the period T_2 of this solution is a submultiple of the assigned period, that is, $pT_2 = qT_1 = T$, where p and q are

* Presented to the Society April 14, 1906. Received for publication June 15, 1906.

† Bulletin Astronomique, vol. 1 (1884), p. 65; Acta Mathematica, vol. 13 (1890), p. 5; *Les méthodes nouvelles de la mécanique céleste*, vol. 1, chap. 3 (1892).

‡ The analysis employed in this paper is valid for sufficiently small values of the parameters involved. All statements concerning the properties of the solutions for $\mu \neq 0$ refer only to values of μ so small that the solutions are known to be valid.

integers. There exists, therefore, a solution of the differential equations for $\mu = 0$, which has the assigned period. Does this solution persist, in the sense of analytic continuation, for values of μ different from zero but sufficiently small? This question is here investigated by the method of POINCARÉ. Three cases present themselves for consideration:

I. When the orbit of the particle is symmetrical with respect to a line.

II. When there exists a uniform integral of the differential equations of motion.

III. When no uniform integral exists.

I. The first case may occur as a particular sub-case of either the second or the third; but, if the differential equations admit a symmetrical solution, the proof of the existence of a symmetrical periodic orbit does not depend upon the consideration of the integral, and may be most conveniently treated as a distinct case.

If a particle is subject to the attraction of two finite bodies, there are always initial conditions under which the orbit is symmetrical with respect to the line joining the finite bodies, and with respect to the time of crossing this line. If the body M_1 revolves about M in a circle, the condition for a symmetrical orbit of the particle P is that P shall cross the radius vector of M_1 orthogonally at any time. The orbits of HILL,* DARWIN,† and MOULTON‡ are of this type. If the orbit of M_1 is an ellipse, the condition is that P shall cross the radius vector orthogonally when M_1 is at an apse.§ The property of the symmetry of the orbit of P with respect to a line rotating with an angular velocity which is a function of t^2, holds for some cases in which the particle is subject to the attraction of more than two finite bodies. When the motion of n finite bodies is such that they form a geometrical figure having always the same shape, in which equal masses are symmetrically situated with respect to a line of the figure, the particle may be started so that its orbit will be symmetrical.‖ An example of another type of the motion of the finite bodies, which is such that the orbit of the particle may be symmetrical, is given in the last section of this paper in the third numerical problem.

II. The case where a uniform integral exists occurs when the finite bodies move in circles about M with the same angular velocity N. The bodies form a fixed geometrical configuration in a plane which rotates about the origin with

* American Journal of Mathematics, vol. 1 (1878), p. 245.

† Acta Mathematica, vol. 21 (1897), p. 99.

‡ Transactions of the American Mathematical Society, vol. 7 (1906), p. 537.

§ This case has been treated in an unpublished paper by W. D. MACMILLAN presented to the Society, April 14, 1906.

‖ Examples are furnished by the equilateral triangular solution of Lagrange, when two of the bodies are equal, and by many particular solutions of the problem of n bodies. See W. R. LONGLEY, *Some particular solutions in the problem of n bodies*, Bulletin of the American Mathematical Society, vol. 13 (1907), p. 324.

the angular velocity N. When the coördinates of the particle are referred to axes in the rotating plane, the differential equations of motion do not contain the time explicitly, and the period of the solution may be chosen arbitrarily except for limitations which are necessary to assure the convergence of the series employed.

III. The third case occurs when the finite bodies do not move in circular orbits about M. The parameter μ is introduced into the differential equations in such a way that the proof of the existence of periodic orbits of the particle is made to depend upon the terms which are due to the attraction of only one of the perturbing bodies, M_1. The motion of M_1 is subject to mild restrictions, namely, that the radius vector shall be given by an expression involving only cosines of multiples of the time, and that the expressions for the longitude shall involve only sines of multiples of the time. The only restriction upon the motion of the other bodies is that it shall be periodic. The treatment is sufficiently general to permit applications in the problems presented by the motions of the solar system. For example, suppose that P is a satellite of one of the planets M, and that M_1 is the sun. The conditions upon the motion of M_1 are fulfilled if we neglect the perturbations of the other planets upon M. If we neglect the inclinations of the orbits of the other planets, and suppose that their motion is periodic, it is possible by the methods given below to treat the periodic motion of the satellite in the plane of the planetary orbit when subject to the attraction of the sun and all the planets.

It is shown that when the period of the motion of the particle is assigned, there exist two and only two real orbits of this analytic type having the prescribed period. In one of the orbits the motion is direct, and in the other it is retrograde. If symmetrical orbits exist, there are no unsymmetrical periodic orbits. It is shown that, in the analysis employed, the eccentricity of the undisturbed orbit must be zero. A method is given for constructing the periodic solutions to any desired degree of accuracy by processes which involve only algebraic computation.

For convenience and clearness the details of the analysis are carried out for a special example of the motion of the finite bodies, namely, for three finite bodies which move according to the equilateral triangular solution of Lagrange. This example illustrates the three cases which may occur, and only slight changes in the details are necessary in order to apply the treatment to the more general examples previously mentioned.

Some numerical examples are given at the end of this paper with drawings of the orbits which have been computed.

Existence of Periodic Orbits.

The differential equations of motion. Let M, M_1, and M_2 denote the masses of three finite bodies moving according to the lagrangian equilateral triangular solution. With reference to M as origin and an axis having a fixed direction in space, let the coördinates of M_1, M_2, and the particle P be respectively (R_1, V_1), $(R_2\ V_2)$, and (r, v). The coördinates of the bodies are expressed in terms of the time as follows:

$$
(1)\qquad
\begin{aligned}
R_1 &= R_2 = AR = A\left\{1 - \epsilon\cos Nt - \frac{\epsilon^2}{2}(\cos 2Nt - 1) + \cdots\right\},\\
V_1 - \varpi_1 &= V_2 - \varpi_2 = V = Nt + 2\epsilon\sin Nt + \tfrac{5}{4}\epsilon^2\sin 2Nt + \cdots,
\end{aligned}
$$

where

$$N^2A^3 = k^2(M + M_1 + M_2).$$

The finite bodies are at apses of their respective orbits at $t = 0$; N is the mean angular motion, A the major semi-axis, ϵ the eccentricity, k is a constant depending upon the units employed; and ϖ_1, ϖ_2 are constants such that $\varpi_2 - \varpi_1 = \pi/3$.

The differential equations of motion of P are

$$
(2)\qquad
\begin{aligned}
&\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{k^2M}{r^2} = \frac{\partial\Omega}{\partial r},\\
&r\frac{d^2v}{dt^2} + 2\frac{dr}{dt}\frac{dv}{dt} = \frac{1}{r}\frac{\partial\Omega}{\partial v},
\end{aligned}
$$

where

$$
(3)\qquad
\begin{aligned}
\Omega &= k^2\left[\frac{M_1}{r_1} + \frac{M_2}{r_2} - \frac{M_1}{R_1^2}r\cos(v - V_1) - \frac{M_2}{R_2^2}r\cos(v - V_2)\right],\\
r_1 &= \sqrt{r^2 + R_1^2 - 2rR_1\cos(v - V_1)},\\
r_2 &= \sqrt{r^2 + R_2^2 - 2rR_2\cos(v - V_2)}.
\end{aligned}
$$

Let us define m and a by the relations

$$(4)\qquad m\nu = N,\qquad \nu^2a^3 = k^2M,$$

where ν is the mean angular velocity of P in the undisturbed orbit.

The period of the motion of the particle in the undisturbed orbit is $2\pi/\nu$. The period of the motion of the finite bodies is $2\pi/N$. In order that the periods shall be commensurable, ν is determined by the relation $2p\pi/\nu = 2q\pi/N$, where p and q are integers. This restriction is not necessary, however, when $\epsilon = 0$.

The next step in obtaining the final form of the differential equations is to refer the motion to a plane rotating with the angular velocity of the finite bodies. If $\epsilon = 0$, the coördinates of M_1 and M_2 become constants, and ν may be

chosen arbitrarily. The motion is referred to an axis rotating with the angular velocity of the finite bodies by the substitution $v = w + V$. It is convenient to eliminate the factors depending upon the units employed by, the substitution

$$r = a\rho, \qquad \nu t = \tau.$$

Hence Nt becomes $m\tau$. We may expand Ω as a power series in $a\rho/AR$ as follows:

$$\begin{aligned}\Omega = \frac{k^2 M_1}{AR}\Bigg[1 &+ \tfrac{1}{4}\left(\frac{a\rho}{AR}\right)^2\Big\{1 + 3\cos 2(w - \varpi_1)\Big\} \\ &+ \tfrac{1}{8}\left(\frac{a\rho}{AR}\right)^3\Big\{3\cos(w - \varpi_1) + 5\cos 3(w - \varpi_1)\Big\} + \cdots\Bigg] \\ + \frac{k^2 M_2}{AR}\Bigg[1 &+ \tfrac{1}{4}\left(\frac{a\rho}{AR}\right)^2\Big\{1 + 3\cos 2(w - \varpi_2)\Big\} \\ &+ \tfrac{1}{8}\left(\frac{a\rho}{AR}\right)^3\Big\{3\cos(w - \varpi_2) + 5\cos 3(w - \varpi_2)\Big\} + \cdots\Bigg]\end{aligned}$$

Let λ_1 and λ_2 be defined by the relations

$$M_1 = \lambda_1(M_1 + M_2), \qquad M_2 = \lambda_2(M_1 + M_2). \tag{5}$$

From equations (1) we have

$$\frac{k^2(M_1 + M_2)}{A^3} = \frac{N^2}{1 + \dfrac{M}{M_1 + M_2}}.$$

On setting

$$\frac{1}{1 + \dfrac{M}{M_1 + M_2}} = K,$$

it follows that

$$\begin{aligned}\frac{1}{a}\frac{\partial\Omega}{\partial(a\rho)} = \frac{Km^2\rho}{R^3}\Bigg[&\frac{\lambda_1}{2}\Big\{1 + 3\cos 2(w - \varpi_1)\Big\} \\ &+ \frac{3\lambda_1}{8}\left(\frac{a\rho}{AR}\right)\Big\{3\cos(w - \varpi_1) + 5\cos 3(w - \varpi_1)\Big\} + \cdots \\ &+ \frac{\lambda_2}{2}\Big\{1 + 3\cos 2(w - \varpi_2)\Big\} \\ &+ \frac{3\lambda_2}{8}\left(\frac{a\rho}{AR}\right)\Big\{3\cos(w - \varpi_2) + 5\cos 3(w - \varpi_2)\Big\} + \cdots\Bigg],\end{aligned} \tag{6}$$

$$
(6)\qquad \begin{aligned}
\frac{1}{a^2\rho}\frac{\partial\Omega}{\partial w} = -\frac{Km^2\rho}{R^3}\Bigg[&\frac{3\lambda_1}{2}\sin 2(w-\varpi_1)\\
&+\frac{3\lambda_1}{8}\left(\frac{a\rho}{AR}\right)\left\{\sin(w-\varpi_1)+5\sin 3(w-\varpi_1)\right\}+\cdots\\
&+\frac{3\lambda_2}{2}\sin 2(w-\varpi_2)\\
&+\frac{3\lambda_2}{8}\left(\frac{a\rho}{AR}\right)\left\{\sin(w-\varpi_2)+5\sin 3(w-\varpi_2)\right\}+\cdots\Bigg].
\end{aligned}
$$

The differential equations of motion become

$$
(7)\qquad \begin{aligned}
&\frac{d^2\rho}{d\tau^2}-\rho\left(\frac{dw}{d\tau}+\frac{dV}{d\tau}\right)^2+\frac{1}{\rho^2}=\frac{1}{a}\frac{\partial\Omega}{\partial(a\rho)},\\
&\rho\left(\frac{d^2w}{d\tau^2}+\frac{d^2V}{d\tau^2}\right)+2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\frac{dV}{d\tau}\right)=\frac{1}{a^2\rho}\frac{\partial\Omega}{\partial w}.
\end{aligned}
$$

Case I. Symmetrical solutions. Suppose the masses M_1 and M_2 are equal and that the w-axis passes through the center of gravity of the system. These conditions are expressed by the relations,

$$
(8)\qquad \lambda_1=\lambda_2,\qquad \varpi_1=-\frac{\pi}{6},\qquad \varpi_2=\frac{\pi}{6}.
$$

It follows that the first of equations (7) involves only *cosines* of multiples of w, and the second involves only *sines* of multiples of w. Since $dV/d\tau$ and R are even functions of τ, it follows that the equations are unchanged if ρ is left unchanged while w and τ are changed into $-w$ and $-\tau$ respectively.* When subject to the restrictions (8), let the differential equations be denoted by (7_{I}).†

Suppose that

$$
(9)\qquad \rho=\psi_1(\tau),\qquad w=\psi_2(\tau),
$$

is a solution of (7_{I}) such that

$$
(10)\qquad \left.\frac{d\rho}{d\tau}\right|_{\tau=p\pi}=0,\qquad w(p\pi)-p\pi=0.
$$

The differential equations are periodic in w and τ, and M_1 and M_2 are at apses at $\tau=p\pi$. It follows from the form of the differential equations that ψ_1 is an even function and ψ_2 is an odd function of τ. Hence if the particle crosses the w-axis when the finite bodies are at apses of their respective orbits, the orbit of P is symmetrical with respect to the axis and the time of crossing. If the orbit is symmetrical with respect to two epochs, for example, $\tau_0=0$ and $\tau_1=p\pi$, it is periodic with the period $2p\pi$. If the solution (9) is symmetrical with

* In the more general cases in which symmetrical solutions occur it may happen that the first equation contains also sines of multiples of w multiplied by odd functions of τ, and that the second equation contains also cosines of multiples of w multiplied by odd functions of τ.

† Throughout this paper the subscripts I, II, III refer to the case I, II, III, respectively.

respect to the epoch $\tau = 0$, the conditions (10) are necessary and sufficient for it to be periodic with the period $2p\pi$.

We will now investigate the existence of symmetrical periodic orbits which have the period $2p\pi$. The terms of equations (7_{I}) are periodic in τ and analytic in m; but, when expanded in this parameter, the coefficients of the various powers of m contain non-periodic terms. It is convenient in this discussion of periodic solutions to avoid such terms in the differential equations, and this is accomplished by a generalization of the parameter m.* Consider the set of differential equations,

$$(11)\qquad \begin{aligned} &\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right)^2 + \frac{1}{\rho^2} = \mu^2 f_{\text{I}}, \\ &\rho\left(\frac{d^2w}{d\tau^2} + \mu\frac{d^2U}{d\tau^2}\right) + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right) = \mu^2 g_{\text{I}}, \end{aligned}$$

where

$$U = \frac{\mu}{m}V = \frac{\mu}{m}\{m\tau + 2\epsilon\sin m\tau + \tfrac{5}{4}\epsilon^2\sin 2m\tau + \cdots\}$$

and $\mu^2 f_{\text{I}}$ and $\mu^2 g_{\text{I}}$ are obtained from $1/a\cdot\partial\Omega/\partial(a\rho)$ and $1/a^2\rho\ \partial\Omega/\partial w$ respectively by writing μ^2 for the factor m^2, and by making the substitution $a/A = \eta\mu$, where η is a numerical constant.

Equations (11) do not represent any physical problem except for a special value of μ, namely, $\mu = m$, and are to be studied from a mathematical point of view. They may be solved for $\rho - 1$ and w as power series in μ. For a preassigned arbitrary interval for τ the series are convergent for all values of μ numerically less than $\bar{\mu}$,† and satisfy the differential equations identically in μ. Therefore, if in the solution μ is given a value numerically equal to $m\,(m < \mu)$, the solution satisfies the differential equations (7_{I}) and represents the physical problem under consideration. It follows from the form in which m enters the differential equations that $\bar{\mu}$ is independent of m.

When $\mu = 0$ the equations are of the form occurring in the problem of two bodies, and a symmetrical solution having the required period is known, namely,

$$(12)\qquad \begin{aligned} &\rho = 1 - \bar{e}\cos E, \\ &w = \operatorname{arc\,cos}\left(\frac{\cos E - \bar{e}}{1 - \bar{e}\cos E}\right) = \operatorname{arc\,sin}\left(\frac{\sqrt{1-\bar{e}^2}\sin E}{1 - \bar{e}\cos E}\right),\ddagger \end{aligned}$$

where E is defined by the relation

$$\tau = E - \bar{e}\sin E,$$

* F. R. Moulton, *On certain rigorous methods of treating problems in celestial mechanics.* The Decennial Publications of The University of Chicago, vol. 3 (1902), p. 126.

† Poincaré, *Les méthodes nouvelles de la mécanique céleste*, vol. 1 (1892), p. 58.

‡ The positive sign is to be taken with the radical.

and $\bar{e}$ is an arbitrary constant less than unity. The initial conditions for $\tau = 0$ are

$$(13) \qquad \rho = 1 - \bar{e}, \qquad \frac{d\rho}{d\tau} = 0, \qquad w = 0, \qquad \frac{dw}{d\tau} = \frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2}.$$

Consider the solution for values of μ different from zero but sufficiently small, and let the initial conditions for $\tau = 0$ be

$$(14) \qquad \begin{aligned} &\rho = 1 - \bar{e} + \beta = (1+\alpha)[1 - (\bar{e} + e)],^* \qquad \frac{d\rho}{d\tau} = 0, \qquad w = 0, \\ &\frac{dw}{d\tau} = \frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2} + \gamma = \frac{\sqrt{1-(\bar{e}+e)^2}}{[1-(\bar{e}+e)]^2} - \mu \left.\frac{dU}{d\tau}\right|_{\tau=0} \end{aligned}$$

The solution is symmetrical with respect to the epoch $\tau = 0$, and may be expressed as power series in α, e, and μ, which are convergent for an interval in τ including the interval 0 to $p\pi$, if the parameters are sufficiently small. If α and e can be determined in terms of μ, vanishing with μ, so that the conditions (10) are satisfied, then the solution will be periodic with the period $2p\pi$. All terms of the solution which are independent of μ^2 may be obtained from the two body problem by making the substitution $w = u - \mu U$.

These terms are given in finite form by the expressions

$$\begin{aligned} &\rho = (1+\alpha) \qquad - (\bar{e} + e)\cos E], \\ &u = \arccos\left(\frac{\cos E - (\bar{e}+e)}{1 - (\bar{e}+e)\cos E}\right) = \arcsin\left(\frac{\sqrt{1-(\bar{e}+e)^2}\sin E}{1-(\bar{e}+e)\cos E}\right), \end{aligned}$$

where E is defined by the relation

$$\frac{\tau}{(1+\alpha)^{\frac{3}{2}}} = E - (\bar{e} + e)\sin E.$$

Returning to the variable w, writing the terms in α and e as power series by Taylor's expansion, and applying the conditions (10), we obtain the equations

$$(15) \qquad \begin{aligned} &0 = -\tfrac{3}{2}p\pi\frac{\bar{e}}{(1-\bar{e})^2}\alpha - \tfrac{3}{4}p\pi\frac{1+\bar{e}}{(1-\bar{e})^3}\alpha e + \cdots, \\ &0 = -\tfrac{3}{2}p\pi\frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2}\alpha - p\pi\mu + \cdots. \end{aligned}$$

It follows from the known properties of the series that there are no terms in e alone, and there are no terms involving μ to the first degree except the term

* The introduction of the very convenient parameters α and e, instead of the additive increments β and γ, is due to Professor F. R. MOULTON, these Transactions, loc. cit., from whom I have received many suggestions and much valuable criticism.

$-p\pi\mu$. Hence the second of equations (15) may be solved for α as a power series in e and μ in which μ is contained as a factor:

$$\alpha = \mu\left\{-\tfrac{2}{3}\frac{(1-\bar{e})^2}{\sqrt{1-\bar{e}^2}} + \cdots\right\}. \tag{16}$$

When this value of α is substituted in the first equation, a factor μ may be divided out, leaving

$$0 = p\pi\frac{\bar{e}}{\sqrt{1-\bar{e}^2}} + \tfrac{1}{2}p\pi\frac{1+\bar{e}}{(1-\bar{e})\sqrt{1-\bar{e}^2}}e + \cdots.$$

This equation may be solved for e as a power series in μ, which vanishes with μ, if and only if $\bar{e}=0$, and the result may then be used to eliminate e from equation (16). Since only those solutions are under consideration which are the analytic continuations with respect to μ of those for $\mu = 0$, the condition $\bar{e} = 0$ must be imposed. *The condition $\bar{e} = 0$ means that the undisturbed orbit must be circular.*

The initial conditions (14) have been determined uniquely as power series in μ, convergent for sufficiently small values of μ, so that the conditions of periodicity are satisfied. When the values of α and e thus found are substituted in the solution, we have $\rho - 1$ and w represented as power series in μ alone, and the solution is periodic with the period $2p\pi$.

Suppose $\mu = m$ and consider the physical interpretation of the solution. The period of the motion of the finite bodies is $2\pi/m = 2p\pi/q$, where p and q are integers which may be chosen arbitrarily except for the condition that q/p must be sufficiently small. The period of the solution is $2p\pi$. Hence during this period the finite bodies make q revolutions, the angle w is increased by $2p\pi$, and therefore the particle makes $q + p$ revolutions with reference to fixed axes. If p and q have the same signs, that is, if m is positive, the motion of the particle is direct. If m is negative, the motion is retrograde. For a given value of m there exists one and only one real symmetrical periodic solution of the differential equations. Hence for a given period there exist two and only two real symmetrical orbits of the particle with the required period. In one the motion is direct, and in the other it is retrograde. All distinct orbits are obtained when p and q are prime to each other.

The initial conditions were limited so that the solutions should be symmetrical. When this condition is not imposed, and whether or not the differential equations admit symmetrical solutions, it will be shown in the next two sections that, for a preassigned period, there exist two and only two real periodic orbits, in one of which the motion is direct and in the other retrograde. Hence, if symmetrical periodic orbits exist, there are no unsymmetrical periodic orbits of this type.

The existence proof was made for a particular example, but did not depend upon the right hand members of the differential equations and consequently holds for any case in which the orbits of the perturbing bodies are such that the differential equations admit symmetrical solutions. The only variations which it is necessary to make in the details for the other cases, are in the particular manner in which the parameter μ is introduced, and in the choice of the rotating axis. It may happen in some cases that there exist several possible axes of symmetry.

Case II. *When a uniform integral exists.* Suppose that M_1 and M_2 are not in general equal and that they move in circles about M with the w-axis passing through M_1. These conditions are expressed by the relations

$$(17) \qquad \epsilon = 0, \qquad \varpi_1 = 0, \qquad \varpi_2 = \frac{\pi}{3}.$$

When subject to the conditions (17) let the differential equations of motion be denoted by (7_{II}).

In this case it is convenient to introduce the parameter μ by the relations

$$(18) \qquad m = \mu, \qquad \lambda_2 = \lambda\mu, \qquad \frac{a}{A} = \eta\mu,$$

for all powers of a/A higher than the *first*. Here it is not necessary to generalize the parameter m since R and $dV/d\tau$ are constants. By relating λ_2 and μ, the existence proof is made to depend only upon those terms of the differential equations which involve λ_1. The generalization of the parameter a/A is merely for convenience in having finite expressions in the equations which determine the coefficients at the various steps in the solution.

The differential equations become

$$(19) \qquad \begin{aligned} &\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = \mu^2 f_{\text{II}}, \\ &\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = \mu^2 g_{\text{II}}, \end{aligned}$$

where

$$f_{\text{II}} = K\rho\left[\lambda_1\left\{\tfrac{1}{2}(1 + 3\cos 2w) + \tfrac{3}{8}\frac{a}{A}\rho(3\cos w + 5\cos 3w)\right\}\right.$$
$$\left. + \cdots \mu \operatorname{func}(\rho, \mu, \cos jw, \sin jw)\right],$$

$$g_{\text{II}} = -K\rho\left[\lambda_1\left\{\tfrac{3}{2}\sin 2w + \tfrac{3}{8}\frac{a}{A}\rho(\sin w + 5\sin 3w)\right\}\right.$$
$$\left. + \cdots \mu \operatorname{func}(\rho, \mu, \sin jw, \cos jw)\right].$$

Equations (19) are periodic in w with period 2π and do not involve τ explicitly. Suppose that

$$\rho = \psi_1(\tau), \qquad w = \psi_2(\tau),$$

is a solution. The necessary and sufficient conditions that the solution shall be periodic with the period $2p\pi$ are

$$(20) \qquad \begin{aligned} &\psi_1(2p\pi) = \psi_1(0), && \left.\frac{d\psi_1}{d\tau}\right|_{2p\pi} = \left.\frac{d\psi_1}{d\tau}\right|_0, \\ &\psi_2(2p\pi) - 2p\pi = \psi_2(0), && \left.\frac{d\psi_2}{d\tau}\right|_{2p\pi} = \left.\frac{d\psi_2}{d\tau}\right|_0. \end{aligned}$$

When $\mu = 0$ a periodic solution is known,* namely,

$$\rho = 1, \qquad w = \tau.$$

The initial conditions for $\tau = 0$ are

$$\rho = 1, \qquad \frac{d\rho}{d\tau} = 0, \qquad w = 0, \qquad \frac{dw}{d\tau} = 1.$$

For μ different from zero, but sufficiently small, consider the solution of equations (19) subject to the initial conditions

$$(21) \qquad \begin{aligned} \rho &= 1 + \beta_1 = (1+\alpha)(1 - e\cos\theta), \\ \frac{d\rho}{d\tau} &= \beta_2 = \frac{e\sin\theta}{\sqrt{1+\alpha}(1-e\cos\theta)}, \\ w &= \beta_3 = \text{arc cos}\left[\frac{\cos\theta - e}{1 - e\cos\theta}\right] - \text{arc cos}\left[\frac{\cos(\theta-\phi) - e}{1 - e\cos(\theta-\phi)}\right], \\ \frac{dw}{d\tau} &= 1 + \beta_4 = \frac{\sqrt{1-e^2}}{(1+\alpha)^{\frac{3}{2}}(1-e\cos\theta)^2} - \mu. \end{aligned}$$

The parameters, β_1, β_2, β_3, β_4, which are additive increments to the initial conditions of undisturbed motion, are inconvenient in the discussion which follows. In terms of the parameters α, e, θ, ϕ, the properties of the solution are well known, and the conditions of periodicity can be easily discussed. The geometrical meaning of the angles θ and ϕ is shown in Fig. 1.

We may now write, in finite form, those terms of the solution which do not contain μ, namely,

$$(22) \qquad \begin{aligned} r &= (1+\alpha)(1 - e\cos E), \\ \cos(w + W_1 - W_0) &= \frac{\cos E - e}{1 - e\cos E}, \\ \sin(w + W_1 - W_0) &= \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}, \end{aligned}$$

* As in the case of symmetrical orbits, no greater generality is obtained by assuming that the undisturbed orbit has an eccentricity different from zero. Therefore we will assume here that $\bar{e} = 0$.

where E is defined by the relation

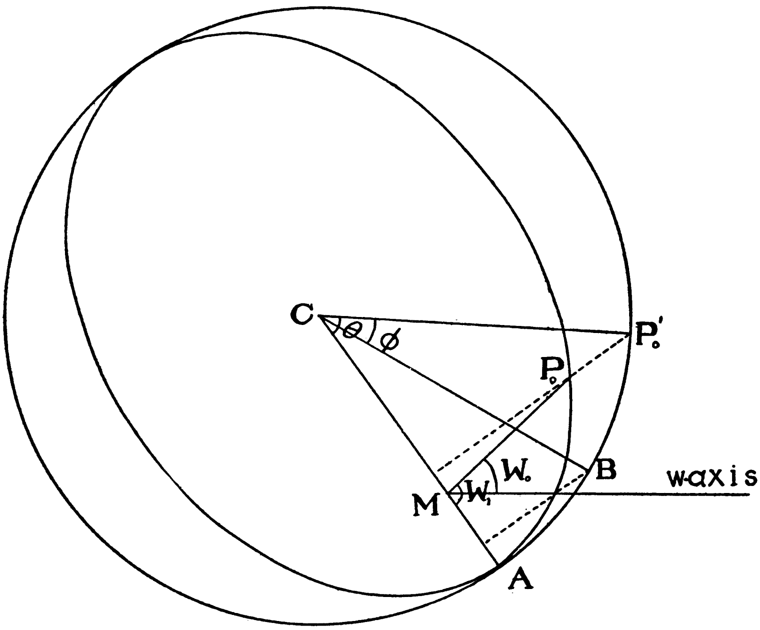
$$\frac{\tau}{(1+\alpha)^{\frac{3}{2}}} + \theta - e \sin \theta = E - e \sin E .$$

The only term of the solution which involves μ to the first degree is $-\tau\mu$, which occurs in the expression for w. To find the terms in μ^2 and $\mu^2\phi$ we may write

$$\rho = 1 + \rho_2 \mu^2 + \bar{\rho}\mu^2\phi + \cdots,$$

$$w = \tau - \tau\mu + \phi + w_2\mu^2 + \bar{w}\mu^2\phi + \cdots.$$

FIG. 1.



P_0 is the position of the particle at $\tau = 0$.
W_0 is the longitude of the particle at $\tau = 0$.
$W_1 - W_0$ is the longitude of perihelion.

On substituting these expressions in the differential equations, there results for the determination of ρ_2 and w_2 the following set of equations,

$$\frac{d^2\rho_2}{d\tau^2} - 2\frac{dw_2}{d\tau} - 3\rho_2 = \frac{K\lambda_1}{2}(1 + 3\cos 2\tau) + \frac{3K\lambda_1}{8}\frac{a}{A}(3\cos\tau + 5\cos 3\tau),$$

$$\frac{d^2 w_2}{d\tau^2} + 2\frac{d\rho_2}{d\tau} = -\frac{3K\lambda_1}{2}\sin 2\tau - \frac{3K\lambda_1}{8}\frac{a}{A}(\sin\tau + 5\sin 3\tau).$$

It follows that

$$\rho_2 = K\lambda_1\left[-1 - 2\frac{a}{A} + \left(2 + \tfrac{153}{64}\frac{a}{A}\right)\cos\tau + \tfrac{15}{16}\frac{a}{A}\tau\sin\tau - \cos 2\tau - \tfrac{25}{64}\frac{a}{A}\cos 3\tau\right],$$

$$w_2 = K\lambda_1\left[\left(\tfrac{5}{4} + 3\frac{a}{A}\right)\tau - \left(4 + \tfrac{201}{32}\frac{a}{A}\right)\sin\tau + \tfrac{15}{8}\frac{a}{A}\tau\cos\tau + \tfrac{11}{8}\sin 2\tau + \tfrac{135}{32}\frac{a}{A}\sin 3\tau\right].$$

By a similar computation it is shown that

$$\bar{\rho} = K\lambda_1\left[-\left(4 + \tfrac{285}{64}\frac{a}{A}\right)\sin\tau + \tfrac{15}{16}\frac{a}{A}\tau\cos\tau + 2\sin 2\tau + \tfrac{75}{64}\frac{a}{A}\sin 3\tau\right],$$

$$\bar{w} = K\lambda_1\left[\tfrac{21}{4} + \tfrac{288}{32}\frac{a}{A} - \left(8 + \tfrac{333}{32}\frac{a}{A}\right)\cos\tau - \tfrac{15}{8}\frac{a}{A}\tau\sin\tau + \tfrac{11}{4}\cos 2\tau + \tfrac{45}{32}\frac{a}{A}\cos 3\tau\right].$$

The terms independent of μ are obtained from equations (22) by Taylor's expansion, and the solution becomes

$$(23)\qquad \begin{aligned} \rho &= 1 + \alpha - e\cos\tau - \alpha e(\cos\tau - \tfrac{3}{2}\tau\sin\tau) + e\theta\sin\tau \\ &\quad + \alpha e\theta(\sin\tau - \tfrac{3}{2}\tau\cos\tau) + \rho_2\mu^2 + \bar{\rho}\mu^2\phi + \cdots, \\ w &= \tau - \tfrac{3}{2}\tau\alpha + 2e\sin\tau + \phi - \tau\mu - 3\alpha e\tau\cos\tau - e\theta(1 - 2\cos\tau) \\ &\quad + e\phi + \alpha e\theta(5\cos\tau + 3\tau\sin\tau) + w_2\mu^2 + \bar{w}\mu^2\phi + \cdots. \end{aligned}$$

Applying the conditions (20) that the solution shall be periodic, we have

$$(24)\qquad \begin{aligned} 0 &= -3p\pi\alpha e\theta + \tfrac{15}{8}\frac{a}{A}p\pi\mu^2\phi + \cdots, &(a)\\ 0 &= 3p\pi\alpha e + \tfrac{15}{8}\frac{a}{A}p\pi\mu^2 + \cdots, &(b)\\ 0 &= -3p\pi\alpha - 2p\pi\mu - 6p\pi\alpha e + \cdots, &(c)\\ 0 &= 6p\pi\alpha e\theta - \tfrac{15}{4}\frac{a}{A}p\pi\mu^2\phi + \cdots. &(d) \end{aligned}$$

The conditions (24) involve the four quantities α, e, θ, ϕ, and, if independent, would determine them in terms of μ. But the differential equations (19) do not involve τ explicitly and hence admit the integral of JACOBI. This fur-

nishes a relation of the type

$$F(\alpha, e, \theta, \phi, \mu) = \text{constant},$$

and equations (24) are not independent.* It follows that if (a), (b), and (c) are solved for the three quantities α, e and θ in terms of μ and ϕ, and the results substituted in (d), the equation is satisfied identically in ϕ. In this problem the dynamical interpretation is simple. Since the finite bodies move in circles the origin of time is arbitrary. The most convenient choice is $\tau = 0$ when $w = 0$, which is equivalent to choosing $\phi = 0$.

Consider the solution of equations (a), (b), and (c) for α, e, and θ. The equations have the following properties:

i. There are no terms independent of α and μ. This follows from the fact that, in the two-body problem, the period does not depend upon e and θ.

ii. There are no terms involving μ to the first degree except the one term $-2p\pi\mu$ which occurs in (c).

iii. There are no terms in θ independent of e, since θ does not enter the initial conditions independently of e.

It follows from these properties and the particular form of the first terms of the equations that α, e, and θ may be determined uniquely as power series in μ by the following steps:

(1) From (c) we obtain

$$\alpha = \mu\left[-\tfrac{2}{3} + \cdots + \text{func}(\mu, e, \theta)\right].$$

(2) This value of α when substituted in (b) permits a factor μ to be divided out. Then we may solve the result for e as a power series in μ and θ, in which μ is contained as a factor,

$$e = \mu\left[\tfrac{15}{16}\,\frac{a}{A} + \cdots + \text{func}(\mu, \theta)\right],$$

(3) When the values of α and e are substituted in (a) a factor μ^2 may be divided out and θ obtained as a power series in μ alone, vanishing with μ.

(4) By substitution of the value of θ thus found in the expressions for e and α, we obtain finally

$$\alpha = \mu\mathfrak{P}_1(\mu), \qquad e = \mu\mathfrak{P}_2(\mu), \qquad \theta = \mu\mathfrak{P}_3(\mu).$$

The preceding operations are known to be convergent for all values of α, e, θ, and μ which are sufficiently small. Hence for a given value of μ sufficiently small it is possible to determine the initial conditions (21) as power series in μ such that the solution of the differential equations (19) shall be periodic in τ with the period $2p\pi$.

Since $\mu\nu = N$, μ is determined by choosing the period of the solution in t

* Cf. POINCARÉ, *Les méthodes nouvelles de la mécanique céleste*, vol. 1 (1892), p. 87.

which is $2p\pi/\nu$. All distinct orbits are obtained by taking $p=1$, that is, every orbit is reëntrant after *one* revolution in the rotating plane. For an assigned period there is one and only one direct orbit, and one and only one retrograde orbit.

The analysis and results apply to the case when any number of finite bodies revolve about M in circles with the same angular velocity.

Case III. *When no uniform integral exists.* Suppose that M_1 and M_2 are not, in general, equal; also, that the eccentricity of the orbits of the finite bodies is not zero, and that the w-axis passes through M_1. These conditions are expressed by the relations

$$\epsilon \neq 0, \qquad \varpi_1 = 0, \qquad \varpi_2 = \frac{\pi}{3}. \tag{25}$$

When subject to the conditions (25), let the differential equations of motion be denoted by (7_{III}). In this case it is convenient to consider the equations

$$\begin{gathered} \frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right)^2 + \frac{1}{\rho^2} = \mu^2 f_{\mathrm{III}}, \\ \rho\left(\frac{d^2w}{d\tau^2} + \mu\frac{d^2U}{d\tau^2}\right) + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right) = \mu^2 g_{\mathrm{III}}, \end{gathered} \tag{26}$$

where $U = \mu/m \cdot V$ and $\mu^2 f_{\mathrm{III}}$ and $\mu^2 g_{\mathrm{III}}$ are obtained from $1/a \cdot \partial\Omega/\partial(a\rho)$ and $1/a^2\rho \cdot \partial\Omega/\partial w$ respectively, by writing μ^2 for the factor m^2 and making the substitution

$$\lambda_2 = \lambda\mu.$$

Equations (20) express the conditions that a solution shall be periodic with the period $2p\pi$. Consider the solution of equations (26) subject to the initial conditions (21), except that the fourth condition contains the term $-\mu \cdot dU/d\tau$ $(\tau = 0)$ instead of $-\mu$. The terms independent of μ are given by equations (22). The only term involving μ to the first degree is $-\mu/m \cdot U$, which occurs in the expression for w.

For the determination of the terms in μ^2 we have the equations

$$\begin{gathered} \frac{d^2\rho_2}{d\tau^2} - 2\frac{dw_2}{d\tau} - 3\rho_2 = \frac{K\lambda_1}{R^3}\left[\tfrac{1}{2}(1+3\cos 2\tau) + \tfrac{3}{8}\frac{a}{AR}(3\cos\tau + 5\cos 3\tau) + \cdots\right], \\ \frac{d^2w_2}{d\tau^2} + 2\frac{d\rho_2}{d\tau} = -\frac{K\lambda_1}{R^3}\left[\tfrac{3}{2}\sin 2\tau + \tfrac{3}{8}\frac{a}{AR}(\sin\tau + 5\sin 3\tau) + \cdots\right]. \end{gathered} \tag{27}$$

R is a power series in ϵ whose coefficients involve only cosines of multiples of $m\tau$. It follows that the right hand member of the first and second equations respectively have the forms

$$A_{0,0} + \sum_{g,h} A_{g,h}\cos(g+hm)\tau, \qquad \sum_{g,h} D_{g,h}\sin(g+hm)\tau,$$

where g and h are integers, g taking all positive values and zero, and h taking positive and zero values such that $hm < 1$. Since τ enters the differential equations explicitly, we have $m = q/p$, by equation (4). $A_{g,h}$ and $D_{g,h}$ are constants depending upon ϵ and a/A; they are infinite series which, for small values of the arguments, converge rapidly.

The second equation of the set (27) may be integrated. The result is *

$$\frac{dw_2}{d\tau} + 2\rho_2 = c_1 - \sum_{g,h} \frac{D_{g,h}}{g + hm} \cos (g + hm)\tau, \tag{28}$$

where c_1 is the constant of integration.

On eliminating $dw_2/d\tau$ between the first of equations (27) and equation (28), there results an equation of the form

$$\frac{d^2 \rho_2}{d\tau^2} + \rho_2 = \bar{A}_{0,0} + \bar{A}_{1,0} \cos \tau + \cdots + \bar{A}_{g,h} \cos (g + hm)\tau + \cdots;$$

whence †

$$\rho_2 = c_2 \cos \tau + c_3 \sin \tau + \bar{A}_{0,0} + \tfrac{1}{2}\bar{A}_{1,0}\tau \sin \tau + \cdots + \frac{A_{g,h}}{1-(g+hm)^2} \cos (g + hm)\tau \cdots,$$

where c_2 and c_3 are constants of integration.

Substituting the value of ρ_2 in equation (28) we get

$$\frac{dw_2}{d\tau} = \bar{D}_{0,0} - 2c_3 \sin \tau - \bar{A}_{1,0}\tau \sin \tau + \cdots + \bar{D}_{g,h} \cos (g + hm)\tau + \cdots;$$

whence w_2 is obtained by quadrature. The constants of integration are determined by the initial conditions,

$$\rho_2 = \frac{d\rho_2}{d\tau} = w_2 = \frac{dw_2}{d\tau} = 0 \qquad (\tau = 0).$$

It is important to observe that all the terms of the solution of equations (27) have the period $2p\pi$, except the terms τ and $\tau \cos \tau$ which occur in w_2, and $\tau \sin \tau$ which occurs in ρ_2. Consequently when the conditions of periodicity (20) are applied, equations (a) and (d) will contain no term in μ^2 alone; (b) and (c) will each contain a term in μ^2 alone.

For the determination of the terms in $\mu^2 \phi$ in the solution we have equations of the form

$$(29) \qquad \begin{aligned} &\frac{d^2\bar{\rho}}{d\tau^2} - 2\frac{d\bar{w}}{d\tau} - 3\bar{\rho} = \sum_{g,h} G_{g,h} \sin (g + hm)\tau, \\ &\frac{d^2\bar{w}}{d\tau^2} + 2\frac{d\bar{\rho}}{d\tau} = H_{0,0} + \sum_{g,h} H_{g,h} \cos (g + hm)\tau. \end{aligned}$$

* By the definition of g, h, and m the denominator $g + hm$ cannot vanish.

† By the definition of g, h, and m the denominator $1 - (g + hm)^2$ cannot vanish.

The solution of the set (29) is of the form

$$\bar{\rho} = 2H_{0,0}\tau - \tfrac{1}{2}\bar{G}_{1,0}\tau\cos\tau + \cdots + \frac{\bar{G}_{g,h}}{1-(g+hm)^2}\sin(g+hm)\tau + \cdots,$$

$$\bar{w} = -\tfrac{3}{2}H_{0,0}\tau^2 + \bar{G}_{1,0}\tau\sin\tau + \cdots - \frac{\bar{H}_{g,h}}{g+hm}\cos(g+hm)\tau + \cdots.$$

The conditions that the solution of equations (26) under consideration shall be periodic with the period $2p\pi$ are

$$(30)\quad \begin{aligned} 0 &= -3p\pi ae\theta + (4H_{0,0}p\pi - \bar{G}_{1,0}p\pi)\mu^2\phi + \cdots, && (a)\\ 0 &= -3p\pi ae + \bar{A}_{1,0}p\pi\mu^2 + \cdots, && (b)\\ 0 &= -3p\pi\alpha + 2p\pi\mu + \cdots, && (c)\\ 0 &= 6p\pi ae\theta - (6H_{0,0}p\pi - 2\bar{G}_{1,0}p\pi)\mu^2\phi + \cdots. && (d)\end{aligned}$$

Equations (30) have the following properties:

i. There are no terms independent of α and μ.
ii. The first power of μ does not occur in (a), (b), and (d).
iii. There are no terms involving ϕ which are independent of μ^2.
iv. There are no terms involving θ which are independent of e.
v. Equations (a), (b), and (d) contain no term in α independent of e.
vi. In equations (a) and (d) there are no terms of degree less than the third.

The knowledge of the preceding properties leads to the conclusion that α, e, θ, and ϕ may be determined uniquely as power series in μ, vanishing with μ and satisfying equations (30). This conclusion is reached by the following steps which are valid for sufficiently small values of α, e, θ, ϕ, and μ.

(1) Let equation (c) be solved for α as a power series in e, θ, ϕ, and μ (which contains μ as a factor by i), giving

$$(31)\qquad \alpha = \tfrac{2}{3}\mu + \mu\mathfrak{P}_c(e, \theta, \phi, \mu).$$

Equation (31) may be used to eliminate α from equations (a), (b), and (d). Let the resulting equations be denoted by (a'), (b'), and (d') respectively.

(2) Then (b') contains a factor μ which may be divided out, leaving

$$0 = -2p\pi e + \bar{A}_{1,0}p\pi\mu + \cdots.$$

This equation contains no term independent of e and μ (by i, ii, and v), and contains a term of the first degree in e. Hence it may be solved for e as a power series in θ, ϕ, and μ, which vanishes with μ,

$$(32)\qquad e = \tfrac{1}{2}\bar{A}_{1,0}\mu + \mu\mathfrak{P}_b(\theta, \phi, \mu).$$

Equation (32) may be used to eliminate e from equations (a') and (d'). Let the resulting equations be denoted by (a'') and (d'') respectively.

(3) Each of equations (a'') and (d'') contains μ^2 as a factor (by i, ii, iii, iv, v, vi), which may be divided out, leaving no term independent of θ, ϕ, and μ:

$$(33)\qquad \begin{aligned} 0 &= -p\pi \bar{A}_{1,0}\theta + (4H_{0,0}p\pi - \bar{G}_{1,0}p\pi)\phi + \cdots, &\qquad (a'')\\ 0 &= 2p\pi \bar{A}_{1,0}\theta - (6H_{0,0}p\pi - 2G_{1,0}p\pi)\phi + \cdots. &\qquad (b'') \end{aligned}$$

The jacobian with respect to θ and ϕ for $\theta = \phi = \mu = 0$ is $-2p^2\pi^2\bar{A}_{1,0}H_{0,0}$, which does not vanish identically in ϵ and a/A. Hence equations (33) may be solved uniquely for θ and ϕ as power series in μ, and the result used to eliminate θ and ϕ from equations (31) and (32). This yields the final solution of of equations (30),

$$(34)\qquad \alpha = \mu\mathfrak{P}_1(\mu), \qquad e = \mu\mathfrak{P}_2(\mu), \qquad \theta = \mu\mathfrak{P}_3(\mu), \qquad \phi = \mu\mathfrak{P}_4(\mu).$$

When these values are substituted in the solution of equations (26), the expressions for ρ and w become power series in μ alone, and the solution is periodic with the period $2p\pi$, since the conditions (30) are satisfied. The physical interpretation is the same as in the preceding cases.

For the purpose of constructing the solutions in applications, it is convenient to have the existence proof in the form given. It leaves open, however, one question which should be answered. That is, are there any values of ϵ different from zero for which the jacobian of equations (33) vanishes? The jacobian is a power series in ϵ and a/A, vanishing with ϵ, and it is not easy to discuss other values of ϵ for which the series might vanish. Such a discussion is made unnecessary by a slight variation in the details of the existence proof. The proof depends only upon the properties of the solution of the problem of two bodies and the non-periodic terms which enter in ρ_2, w_2, $\bar{\rho}$, and $\bar{w}$, and may be carried through without considering in equations (27) and (29) the infinite series in ϵ which represents $1/R$ and the infinite series in a/A. This is accomplished by picking out from f_{III} in (26) a single term of the type

$$\epsilon^p\left(\frac{a}{A}\right)^{q-1}\cos pm\tau \cos(q-1)w$$

and from g_{III} a single term of the type

$$\epsilon^p\left(\frac{a}{A}\right)^{q-1}\cos pm\tau \sin(q-1)w.$$

These terms are left unchanged, while in all other terms ϵ is replaced by $\sigma\mu$ and a/A by $\eta\mu$. The set of differential equations thus obtained may be treated by steps similar to those employed in the consideration of equations (26), and a set

of equations corresponding to (33) will be obtained. The jacobian of this set is $\epsilon^{2p}(a/A)^{2q-2}$, except for a constant factor, and can vanish only if $\epsilon = 0$.

Construction of the Solutions.

It has been demonstrated for each of the three sets of differential equations (11), (19), and (26) that, for a given value of μ sufficiently small, it is possible to determine the initial conditions uniquely as power series in μ such that the solution as power series in μ shall be periodic in τ with the period $2p\pi$. A method will now be given by which the solution to any desired number of terms may be conveniently constructed. It is not necessary to determine the initial conditions explicitly in advance and the computation involves only algebraic processes. It will be convenient for brevity in the treatment to consider first

Case III. *When no uniform integral exists.* Consider equations (26), and let the periodic solution, which is known to exist, be written

$$
\begin{aligned}
\rho - 1 &= \rho_1\mu + \rho_2\mu^2 + \rho_3\mu^3 + \cdots + \rho_i\mu^i + \cdots, \\
w - \tau &= w_1\mu + w_2\mu^2 + w_3\mu^3 + \cdots + w_i\mu^i + \cdots.
\end{aligned} \tag{35}
$$

Since the solution exists and is periodic for all values of μ sufficiently small, each coefficient is periodic with the required period.

Let the solution (35) be substituted into the differential equations (26) and the result arranged as power series in μ. The terms of the left-hand members have the following forms. (The accents indicate derivatives with respect to τ.)

$$U = \tau + \bar{U} = \tau + \frac{2}{m}\epsilon\sin m\tau + \frac{5}{4m}\epsilon^2\sin 2m\tau + \cdots,$$

$$\frac{dU}{d\tau} = 1 + \bar{U}', \qquad \frac{d^2U}{d\tau^2} = \bar{U}'',$$

$$\frac{d^2\rho}{d\tau^2} = \rho_1''\mu + \rho_2''\mu^2 + \rho_3''\mu^3 + \cdots + \rho_i''\mu^i + \cdots,$$

$$
\begin{aligned}
\rho\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right)^2 = {} & 1 + [\rho_1 + 2(w_1' + 1 + \bar{U}')]\mu \\
& + [\rho_2 + 2w_2' + 2\rho_1(w_1' + 1 + \bar{U}') + (w_1' + 1 + \bar{U}')^2]\mu^2 \\
& + [\rho_3 + 2w_3' + 2(w_1' + 1 + \bar{U}')w_2' + 2\rho_1 w_2' + 2\rho_2(w_1' + 1 + \bar{U}') \\
& + \rho_1(w_1 + 1 + \bar{U}')^2]\mu^3 + \cdots + [\rho_i + 2w_i' + 2(w_1' + 1 + \bar{U}')w_{i-1}' \\
& + 2\rho_{i-1}(w_1' + 1 + \bar{U}') + 2\rho_1 w_{i-1}' + \cdots]\mu^i + \cdots,
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\frac{1}{\rho^2} = {} & 1 - 2\rho_1\mu - (2\rho_2 - 3\rho_1^2)\mu^2 - (2\rho_3 - 6\rho_1\rho_2 + 4\rho_1^3)\mu^3 \\
& + \cdots + (2\rho_i - 6\rho_{i-1}\rho_1 + \cdots)\mu^i + \cdots,
\end{aligned}
$$

$$\rho\left(\frac{d^2 w}{d\tau^2}+\mu\frac{d^2 U}{d\tau^2}\right)=(w_1''+\bar{U}'')\mu+[w_2''+\rho_1(w_1''+\bar{U}'')]\mu^2$$
$$+[w_3''+w_2''\rho_1+\rho_2(w_1''+\bar{U}'')]\mu^3+\cdots$$
$$+[w_i''+w_{i-1}''\rho_1+\cdots+\rho_{i-1}(w_1''+\bar{U}'')]\mu^i+\cdots,$$
$$\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\frac{dU}{d\tau}\right)=\rho_1'\mu+[\rho_2'+\rho_1'(w_1'+1+\bar{U}')]\mu^2$$
$$+[\rho_3'+\rho_2'(w_1'+1+\bar{U}')+\rho_1'w_2']\mu^3+\cdots$$
$$+[\rho_i'+\rho_{i-1}'(w_1'+1+\bar{U}')+\cdots+\rho_1'w_{i-1}']\mu^i+\cdots.$$

The equations hold identically in μ. Hence, equating the coefficients of the first power of μ, we have for the determination of ρ_1 and w_1 the following set of equations:

$$(37)\qquad \frac{d^2\rho_1}{d\tau^2}-2\frac{dw_1}{d\tau}-3\rho_1=2(1+\bar{U}'),\qquad \frac{d^2 w_1}{d\tau^2}+2\frac{d\rho_1}{d\tau^2}=-\bar{U}''.$$

It follows that

$$(38)\qquad \begin{aligned}\rho_1&=2(1+c_1^{(1)})+c_2^{(1)}\cos\tau+c_3^{(1)}\sin\tau,\\ w_1&=c_4^{(1)}-(4+3c_1^{(1)})\tau-\bar{U}-2c_2^{(1)}\sin\tau+2c_3^{(1)}\cos\tau,\end{aligned}$$

where $c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$, and $c_4^{(1)}$ are constants of integration. Since ρ_1 and w_1 are periodic, the coefficient of τ in w_1 must vanish. This condition determines the constant $c_1^{(1)}$, namely $c_1^{(1)}=-\frac{4}{3}$. The other constants of integration are so far arbitrary.

On equating the coefficients of the second power of μ, the following set of equations is obtained:

$$(39)\qquad \begin{aligned}\frac{d^2\rho_2}{d\tau^2}-2\frac{dw_2}{d\tau}-3\rho_2&=(w_1'+1+\bar{U}')^2+2\rho_1(w_1'+1+\bar{U}')-3\rho_1^2+f_{\mathrm{III},0},\\ \frac{d^2 w_2}{d\tau^2}+2\frac{d\rho_2}{d\tau}&=-\rho_1(w_1''+\bar{U}'')-2\rho_1'(w_1'+1+\bar{U}')+g_{\mathrm{III},0},\end{aligned}$$

where $f_{\mathrm{III},0}$ and $g_{\mathrm{III},0}$ are obtained from f_{III} and g_{III} respectively by writing $\mu=0$, $w=\tau$, $\rho=1$. The right-hand members are known functions of τ and the equations have the form

$$(40)\qquad \begin{aligned}\frac{d^2\rho_2}{d\tau^2}-2\frac{dw_2}{d\tau}-3\rho_2&=A_{0,0}^{(2)}+(\tfrac{14}{3}c_2^{(1)}+A_{1,0}^{(2)})\cos\tau+\tfrac{14}{3}c_3^{(1)}\sin\tau\\ &\quad+\cdots+A_{g,h}^{(2)}\cos(g+hm)\tau+\cdots,\\ \frac{d^2 w_2}{d\tau^2}+2\frac{d\rho_2}{d\tau}&=(\tfrac{10}{3}c_2^{(1)}+D_{1,0}^{(2)})\sin\tau-\tfrac{10}{3}c_3^{(1)}\cos\tau\\ &\quad+\cdots+D_{g,h}^{(2)}\sin(g+hm)\tau+\cdots.\end{aligned}$$

It is to be noted that $c_4^{(1)}$ does not occur in equations (40). Integrating the second equation we have

$$\frac{dw_2}{d\tau} + 2\rho_2 = c_1^{(2)} - (\tfrac{10}{3}c_2^{(1)} + D_{1,0}^{(2)})\cos\tau - \tfrac{10}{3}c_3^{(1)}\sin\tau + \cdots - \frac{D_{g,h}^{(2)}}{g+hm}\cos(g+hm)\tau + \cdots. \tag{41}$$

On eliminating $dw_2/d\tau$ from the first of equations (40) by means of equation (41), there results

$$\frac{d^2\rho_2}{d\tau^2} + \rho_2 = A_{0,0}^{(2)} + 2c_1^{(2)} + [-2c_2^{(1)} + A_{1,0}^{(2)} - 2D_{1,0}^{(2)}]\cos\tau - 2c_3^{(1)}\sin\tau + \cdots + \left(A_{g,h}^{(2)} - \frac{2D_{g,h}^{(2)}}{g+hm}\right)\cos(g+hm)\tau + \cdots. \tag{42}$$

In order that the solution of equation (42) shall contain no non-periodic term, the coefficients of $\cos\tau$ and $\sin\tau$ must vanish, whence $c_2^{(1)}$ and $c_3^{(1)}$ are determined by the conditions

$$2c_2^{(1)} = A_{1,0}^{(2)} - 2D_{1,0}^{(2)}, \qquad c_3^{(1)} = 0.$$

With these values of $c_2^{(1)}$ and $c_3^{(1)}$, the solution becomes

$$\rho_2 = A_{0,0}^{(2)} + 2c_1^{(2)} + c_2^{(2)}\cos\tau + c_3^{(2)}\sin\tau + \cdots + \alpha_{g,h}^{(2)}\cos(g+hm)\tau + \cdots,$$

where

$$\alpha_{g,h}^{(2)} = \frac{1}{1-(g+hm)^2}\left(A_{g,h}^{(2)} - \frac{2D_{g,h}^{(2)}}{g+hm}\right).$$

Substituting this value of ρ_2 in equation (41) and integrating, we obtain for w_2 a solution of the form

$$w_2 = c_4^{(2)} - (2A_{0,0}^{(2)} + 3c_1^{(2)})\tau - 2c_2^{(2)}\sin\tau + 2c_3^{(2)}\cos\tau + \cdots + \delta_{g,h}^{(2)}\sin(g+hm)\tau + \cdots,$$

where

$$\delta_{g,h}^{(2)} = -\frac{1}{g+hm}\left(\frac{D_{g,h}^{(2)}}{g+hm} + 2\alpha_{g,h}^{(2)}\right).$$

Since w_2 is periodic, $c_1^{(2)}$ is determined by the condition

$$2A_{0,0}^{(2)} + 3c_1^{(2)} = 0.$$

Of the eight constants of integration, which have been introduced in the first two steps, four ($c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$, $c_1^{(2)}$) have been determined uniquely and four ($c_4^{(1)}$, $c_2^{(2)}$, $c_3^{(2)}$, $c_4^{(2)}$) are still arbitrary.

By equating the coefficients of the third power of μ the following set of equations is obtained:

$$\begin{aligned}\frac{d^2\rho_3}{d\tau^2} - 2\frac{dw_3}{d\tau} - 3\rho_3 &= 2(w_1' + 1 + \bar{U}' + \rho_1)w_2' + 2\rho_2(w_1' + 1 + \bar{U}') \\ &\quad + \rho_1(w_1' + 1 + \bar{U}')^2 - 6\rho_1\rho_2 + 4\rho_1^3 + f_{\mathrm{III},1}, \\ \frac{d^2 w_3}{d\tau^2} + 2\frac{d\rho_3}{d\tau} &= -\rho_1 w_2'' - (w_1'' + \bar{U}'')\rho_2 - 2\rho_2'(w_1' + 1 + \bar{U}') - 2\rho_1' w_2' + g_{\mathrm{III},1},\end{aligned} \tag{43}$$

where $f_{\mathrm{III},1}$ and $g_{\mathrm{III},1}$ denote the coefficients of μ in f_{III} and g_{III} respectively. Equations (43) do not contain $c_4^{(2)}$. The right-hand members contain $c_4^{(1)}$ linearly. The equations have the form

$$
(44)\quad
\begin{aligned}
\frac{d^2\rho_3}{d\tau^2} - 2\frac{dw_3}{d\tau} - 3\rho_3 &= A^{(3)}_{0,0} + (\tfrac{14}{3}c_2^{(2)} + A^{(3)}_{1,0})\cos\tau + (\tfrac{14}{3}c_3^{(2)} + B^{(3)}_{1,0})\sin\tau \\
&\quad + \cdots + A^{(3)}_{g,h}\cos(g+hm)\tau + B^{(3)}_{g,h}\sin(g+hm)\tau + \cdots, \\
\frac{d^2w_3}{d\tau^2} + 2\frac{d\rho_3}{d\tau} &= C^{(3)}_{0,0} + (\tfrac{10}{3}c_2^{(2)} + D^{(3)}_{1,0})\sin\tau + (-\tfrac{10}{3}c_3^{(2)} + C^{(3)}_{1,0})\cos\tau \\
&\quad + \cdots D^{(3)}_{g,h}\sin(g+hm)\tau + C^{(3)}_{g,h}\cos(g+hm)\tau + \cdots.
\end{aligned}
$$

Here $C^{(3)}_{0,0}$ involves $c_4^{(1)}$ linearly. To avoid non-periodic terms in the solution, $C^{(3)}_{0,0}$ must vanish, and this condition determines $c_4^{(1)}$ uniquely, if the coefficient of $c_4^{(1)}$ does not vanish. The coefficient is a power series in ϵ and a/A which vanishes with ϵ but not identically in ϵ. The question concerning special values of ϵ for which it might vanish is the same as the question concerning the vanishing of the jacobian of equations (33) and is answered in the same way.

The treatment of equations (44) proceeds by steps similar to those employed in the solution of equations (40). Four new constants of integration are introduced, namely, $c_1^{(3)}$, $c_2^{(3)}$, $c_3^{(3)}$, $c_4^{(3)}$; while $c_2^{(2)}$, $c_3^{(2)}$, and $c_1^{(3)}$ are uniquely determined by the conditions:

$$
2c_2^{(2)} = A^{(3)}_{1,0} - 2D^{(3)}_{1,0}, \qquad 2c_3^{(2)} = B^{(3)}_{1,0} + 2C^{(3)}_{1,0}, \qquad 3c_1^{(3)} = -2A^{(3)}_{0,0}.
$$

So far $c_4^{(2)}$, $c_2^{(3)}$, $c_3^{(3)}$, and $c_4^{(3)}$ are arbitrary.

It may be established by complete induction that the above process can be carried as far as is desired. Suppose $\rho_1, w_1;\ \rho_2, w_2;\ \cdots \rho_{i-1}, w_{i-1}$ have been determined by this process. The expressions have the following form

$$
\begin{aligned}
\rho_l &= \alpha^{(l)}_{0,0} + \cdots + \alpha^{(l)}_{g,h}\cos(g+hm)\tau + \beta^{(l)}_{g,h}\sin(g+hm)\tau + \cdots, \\
w_l &= \gamma^{(l)}_{0,0} + \cdots + \delta^{(l)}_{g,h}\sin(g+hm)\tau + \gamma^{(l)}_{g,h}\cos(g+hm)\tau + \cdots,
\end{aligned}
\qquad (l = 1, 2, \cdots, i-3).
$$

$$
\begin{aligned}
\rho_{i-2} &= \alpha^{(i-2)}_{0,0} + \cdots + \alpha^{(i-2)}_{g,h}\cos(g+hm)\tau + \beta^{(i-2)}_{g,h}\sin(g+hm)\tau + \cdots, \\
w_{i-2} &= c_4^{(i-2)} + \cdots + \delta^{(i-2)}_{g,h}\sin(g+hm)\tau + \gamma^{(i-2)}_{g,h}\cos(g+hm)\tau + \cdots.
\end{aligned}
$$

$$
\begin{aligned}
\rho_{i-1} &= \alpha^{(i-1)}_{0,0} + c_2^{(i-1)}\cos\tau + c_3^{(i-1)}\sin\tau + \cdots, \\
w_{i-1} &= c_4^{(i-1)} - 2c_2^{(i-1)}\sin\tau + 2c_3^{(i-1)}\cos\tau + \cdots.
\end{aligned}
$$

The equations for the determination of ρ_i and w_i are

$$
(45)\quad
\begin{aligned}
\frac{d^2\rho_i}{d\tau^2} - 2\frac{dw_i}{d\tau} - 3\rho_i &= A^{(i)}_{0,0} + (\tfrac{14}{3}c_2^{(i-1)} + A^{(i)}_{1,0})\cos\tau + (\tfrac{14}{3}c_3^{(i-1)} + B^{(i)}_{1,0})\sin\tau \\
&\quad + \cdots + A^{(i)}_{g,h}\cos(g+hm)\tau + B^{(i)}_{g,h}\sin(g+hm)\tau + \cdots, \\
\frac{d^2w_i}{d\tau^2} + 2\frac{d\rho_i}{d\tau} &= C^{(i)}_{0,0} + (\tfrac{10}{3}c_2^{(i-1)} + D^{(i)}_{1,0})\sin\tau + (-\tfrac{10}{3}c_3^{(i-1)} + C^{(i)}_{1,0})\cos\tau \\
&\quad + \cdots + D^{(i)}_{g,h}\sin(g+hm)\tau + C^{(i)}_{g,h}\cos(g+hm)\sin\tau + \cdots.
\end{aligned}
$$

Equations (45) do not contain $c_4^{(i-1)}$. The coefficient of $c_4^{(i-2)}$ in $C_{0,0}^{(i)}$ is the same as the coefficient of $c_4^{(1)}$ in $C_{0,0}^{(3)}$. Equations (45) are solved by the steps employed for the solution of equations (40). During the process four constants of integration are introduced, namely, $c_1^{(i)}, c_2^{(i)}, c_3^{(i)}, c_4^{(i)}$, and four are uniquely determined by the following conditions, the first of which determines $c_4^{(i-2)}$;

$$(46)\qquad \begin{aligned} &C_{0,0}^{(i)} = 0, & 2c_2^{(i-1)} &= A_{1,0}^{(i)} - 2D_{1,0}^{(i)}, \\ &2c_3^{(i-1)} = B_{1,0}^{(i)} + 2C_{1,0}^{(i)}, & 3c_1^{(i)} &= -2A_{0,0}^{(i)}. \end{aligned}$$

The solution of equations (45) is

$$(47)\qquad \begin{aligned} \rho_i &= \alpha_{0,0}^{(i)} + c_2^{(i)}\cos\tau + c_3^{(i)}\sin\tau + \cdots \\ &\qquad + \alpha_{g,h}^{(i)}\cos(g+hm)\tau + \beta_{g,h}^{(i)}\sin(g+hm)\tau + \cdots, \\ w_i &= c_4^{(i)} + \delta_{1,0}^{(i)}\sin\tau + \gamma_{1,0}^{(i)}\cos\tau + \cdots \\ &\qquad + \delta_{g,h}^{(i)}\sin(g+hm)\tau + \gamma_{g,h}^{(i)}\cos(g+hm)\tau + \cdots, \end{aligned}$$

where the coefficients are given by the formulas:

$$(48)\qquad \begin{aligned} \alpha_{0,0}^{(i)} &= -\tfrac{1}{3}A_{0,0}^{(i)}, \\ \alpha_{g,h}^{(i)} &= \frac{1}{1-(g+hm)^2}\left(A_{g,h}^{(i)} - \frac{2D_{g,h}^{(i)}}{g+hm}\right), \\ \beta_{g,h}^{(i)} &= \frac{1}{1-(g+hm)^2}\left(B_{g,h}^{(i)} + \frac{2C_{g,h}^{(i)}}{g+hm}\right), \\ \delta_{1,0}^{(i)} &= -2c_2^{(i)} - \tfrac{5}{3}A_{1,0}^{(i)} + \tfrac{7}{3}D_{1,0}^{(i)}, \\ \delta_{g,h}^{(i)} &= -\frac{1}{g+hm}\left(\frac{D_{g,h}^{(i)}}{g+hm} + 2\alpha_{g,h}^{(i)}\right), \\ \gamma_{1,0}^{(i)} &= 2c_3^{(i)} + \tfrac{5}{3}B_{1,0}^{(i)} + \tfrac{7}{3}C_{1,0}^{(i)}, \\ \gamma_{g,h}^{(i)} &= -\frac{1}{g+hm}\left(\frac{C_{g,h}^{(i)}}{g+hm} - 2\beta_{g,h}^{(i)}\right). \end{aligned}$$

This completes the proof of the statement that the solution may be constructed to any desired degree of accuracy. Of the four constants of integration which occur in the calculation of the coefficients of μ^i, one, $c_1^{(i)}$, is determined in the step in which it enters; two more, $c_2^{(i)}$ and $c_3^{(i)}$, are determined in the next following step; while the fourth, $c_4^{(i)}$, is determined in the second following step.

Case II. *When a uniform integral exists.* The method of constructing the solution in this case differs from the preceding in one particular. The general steps are the same and formulas (48) are applicable. The difference occurs in the determination of the constant $c_4^{(i-2)}$. In the proof of the existence of a

periodic solution of equations (19) it was shown that the origin of time is arbitrary, which is equivalent to the statement that at $\tau = \tau_0$ the value $w = w_0$ may be assumed arbitrarily. It follows that $C_{0,0}^{(i)} = 0$ and does not determine $c_4^{(i-2)}$. This constant is determined by assuming an initial condition, for example, $w = 0$ at $\tau = 0$. Then $c_4^{(i-2)}$ is determined in the step in which it enters by the condition $w_{i-2} = 0$ at $\tau = 0$. When the step for computing the coefficients of μ^i is completed, there remain two constants of integration, $c_2^{(i)}$ and $c_3^{(i)}$ which have not been determined. These constants are determined in the next following step.

Case I. *Symmetrical solutions.* In this case there are two arbitrary initial conditions and the method of constructing the solution differs from that of case III in two particulars. The differential equations are (11). Consider the first step in the construction [equations (37) and (38)]. We determine $c_3^{(1)}$ and $c_4^{(1)}$ respectively by the conditions

$$\frac{d\rho_1}{d\tau} = 0, \qquad w_1 = 0 \qquad (\tau = 0).$$

Therefore $c_3^{(1)} = c_4^{(1)} = 0$. And $c_1^{(1)}$ is determined by the condition that w_1 shall be periodic while $c_2^{(1)}$ is not determined until the next step.

This process is applicable to all the following steps. The differential equations (11) have a particular form which admits a symmetrical solution, and it may be established by complete induction that the equations for the determination of ρ_i and w_i have the form

$$\text{(49)}\quad \begin{aligned} &\frac{d^2\rho_i}{d\tau^2} - 2\frac{dw_i}{d\tau} - 3\rho_i = A_{0,0}^{(i)} + (\tfrac{14}{3}c_2^{(i-1)} + A_{1,0}^{(i)})\cos\tau + \cdots + A_{g,h}^{(i)}\cos(g+hm)\tau + \cdots, \\ &\frac{d^2 w_i}{d\tau^2} + 2\frac{d\rho_i}{d\tau} = (\tfrac{10}{3}c_2^{(i-1)} + D_{1,0}^{(i)})\sin\tau + \cdots + D_{g,h}^{(i)}\sin(g+hm)\tau + \cdots. \end{aligned}$$

where $c_2^{(i-1)}$ is determined by the condition

$$2c_2^{(i-1)} = A_{1,0}^{(i)} - 2D_{1,0}^{(i)}.$$

The solution of (49) is

$$\rho_i = \alpha_{0,0}^{(i)} + c_1^{(i)}\cos\tau + \cdots + \alpha_{g,h}^{(i)}\cos(g+hm)\tau + \cdots,$$

$$w_i = \delta_{1,0}^{(i)}\sin\tau + \cdots + \delta_{g,h}^{(i)}\sin(g+hm)\tau + \cdots,$$

where

$$\alpha_{0,0}^{(i)} = -\tfrac{1}{3}A_{0,0}^{(i)},$$

$$\alpha_{g,h}^{(i)} = \frac{1}{1-(g+hm)^2}\left(A_{g,h}^{(i)} - \frac{2D_{g,h}^{(i)}}{g+hm}\right),$$

$$\delta_{1,0}^{(i)} = -2c_2^{(i)} - \tfrac{5}{3}A_{1,0}^{(i)} + \tfrac{7}{3}D_{1,0}^{(i)},$$

$$\delta_{g,h}^{(i)} = -\frac{1}{g+hm}\left(\frac{D_{g,h}^{(i)}}{g+hm} + 2\alpha_{g,h}^{(i)}\right).$$

Numerical Examples.

As illustrations of the preceding analysis three numerical examples will be treated briefly. It has not been shown that the processes are valid for the numerical values which are employed and which have been selected for convenience in graphical representation. It is probable that the series are convergent, although it has not been found possible to determine the true radii of convergence.

In each case the mass of M is taken as the unit of mass, and M_1, of mass 10, is supposed to move about M in a circle whose radius is taken as the unit of distance. The unit of time is chosen so that the angular velocity N of M_1 is unity. The orbits considered are those in which the particle revolves about M. The period is chosen so that $\nu = 5$, whence $\mu = 0.2$.

In the first example M_2 of mass 5 is supposed to revolve about M in a circle so that the three finite masses remain always in a straight line in the order $M_2 M M_1$. It follows from the solution of the quintic equation of LAGRANGE that the distance $M_2 M$ is $0.77172 \cdots$. The orbit of the particle is symmetrical with respect to the line joining the bodies.

In the second example M_2 of mass 5 is supposed to revolve about M in a circle so that the three bodies remain always at the vertices of an equilateral triangle. The differential equations admit the integral of JACOBI, but there is no symmetry theorem.

In the third case it is supposed that M_2 of unit mass revolves about M_1 in a circle of radius A_2. The orbit of the particle is symmetrical with respect to the radius vector of M_1 and crosses this line when the finite bodies are in conjunction.

The differential equations of motion. With reference to M as origin and an axis passing always through M_1 let the coördinates of M_1, M_2, and P be respectively $(1, 0)$, (R_2, W_2), and (r, w). The equations of motion, see equations (7), of the particle are

$$
\begin{aligned}
\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = {} & k^2 M_1 \mu^2 \rho \left[\tfrac{1}{2}\{1 + 3\cos 2w\} + \tfrac{3}{8} a\rho\{3\cos w + 5\cos 3w\}\right. \\
& \left. + \tfrac{1}{16} a^2\rho^2\{9 + 20\cos 2w + 35\cos 4w\} + \cdots\right] \\
& + \frac{k^2 M_2}{R_2^3}\mu^2\rho\left[\tfrac{1}{2}\{1 + 3\cos 2(w - W_2)\} + \tfrac{3}{8}\frac{a}{R^2}\rho\{3\cos(w - W_2) + 5\cos 3(w - W_2)\}\right. \\
& \left. + \tfrac{1}{16}\left(\frac{a}{R_2}\right)^2\rho^2\{9 + 20\cos 2(w - W_2) + 35\cos 4(w - W_2)\} + \cdots\right],
\end{aligned}
$$

$$
\begin{aligned}
\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = {} & -k^2 M_1 \mu^2 \rho\left[\tfrac{3}{2}\sin 2w + \tfrac{3}{8} a\rho\{\sin w + 5\sin 3w\}\right. \\
& \left. + \tfrac{5}{16} a^2\rho^2\{2\sin 2w + 7\sin 4w\} + \cdots\right] \\
& - \frac{k^2 M_2}{R_2^3}\mu^2\rho\left[\tfrac{3}{2}\sin 2(w - W_2) + \tfrac{3}{8}\frac{a}{R_2}\rho\{\sin(w - W_2) + 5\sin 3(w - W_2)\}\right. \\
& \left. + \tfrac{5}{16}\left(\frac{a}{R_2}\right)^2\rho^2\{2\sin 2(w - W_2) + 7\sin 4(w - W_2)\} + \cdots\right],
\end{aligned}
$$

where a is given by the relation $\nu^2 a^3 = k^2$. The numerical value of k^2 depends upon the masses and orbits of the finite bodies and must be determined separately for each of the three examples.

Example 1. In this case $R_2 = 0.77172$ and $W_2 = \pi$. Now k^2 is given by the relation

$$N^2 = \frac{M_1 + M + M_2}{M + M_2 + M_2 R_2}\left(M + \frac{M_2}{(1 + R_2^2)}\right) k^2,$$

whence

$$k^2 = 0.23763.$$

It follows that

$$k^2 M_1 = 2.37630, \qquad \frac{k^2 M_2}{R_2^3} = 2.58518,$$

$$a = 1.05914\mu, \qquad \frac{a}{R_2} = 1.37222\mu.$$

The equations of motion become

$$(50)\quad \begin{aligned} \frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} &= (2.48074 + 7.44222\cos 2w)\rho\mu^2 \\ &\quad - (1.15938\cos w + 1.93230\cos 3w)\rho^2\mu^3 \\ &\quad + (4.23765 + 9.41700\cos 2w + 16.47975\cos 4w)\rho^3\mu^4 + \cdots, \\ \rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) &= -(7.44222\sin 2w)\rho\mu^2 \\ &\quad + (0.38646\sin w + 1.93230\sin 3w)\rho^2\mu^3 \\ &\quad - (4.70850\sin 2w + 16.47975\sin 4w)\rho^3\mu^4 + \cdots. \end{aligned}$$

The periodic solution of equations (50) is

$$(51)\quad \begin{aligned} \rho &= 1 - \tfrac{2}{3}\mu - (0.27136 + 0.96615\cos\tau + 4.96148\cos 2\tau)\mu^2 \\ &\quad + (0.62584 - 19.13740\cos\tau - 2.47963\cos 2\tau + 0.40256\cos 3\tau)\mu^3 + \cdots, \\ w &= \tau + (1.93230\sin\tau + 6.82204\sin 2\tau)\mu^2 \\ &\quad + (41.10885\sin\tau + 10.74793\sin 2\tau - 0.48307\sin 3\tau)\mu^3 + \cdots. \end{aligned}$$

Example 2. In this case $R_2 = 1$ and $W_2 = \pi/3$. Here k^2 is given by the relation

$$N^2 = k^2(M + M_1 + M_2),$$

whence

$$k^2 = 0.06250.$$

It follows that

$$k^2 M_1 = 0.62500, \qquad k^2 M_2 = 1.56250\mu, \qquad a = 0.67860\mu.$$

The equations of motion become

$$(52)\quad \begin{aligned} \frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} &= (0.31250 + 0.93750 \cos 2w)\rho\mu^2 \\ &+ (0.78125 - 1.17188 \cos 2w + 2.02977 \sin 2w)\rho\mu^3 \\ &+ (0.47714 \cos w + 0.79523 \cos 3w)\rho^2\mu^3 \\ &+ (0.59643 \cos w + 1.03308 \sin w - 1.98808 \cos 3w)\rho^2\mu^4 \\ &+ (0.16188 + 0.35974 \cos 2w + 0.62954 \cos 4w)\rho^3\mu^4 + \cdots, \\ \rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) &= -(0.93750 \sin 2w)\rho\mu^2 \\ &+ (1.17188 \sin 2w + 2.02977 \cos 2w)\rho\mu^3 \\ &- (0.15905 \sin w + 0.79523 \sin 3w)\rho^2\mu^3 \\ &+ (-0.19881 \sin w + 0.34436 \cos w + 1.98808 \sin 3w)\rho^2\mu^4 \\ &- (0.17987 \sin 2w + 0.62954 \sin 4w)\rho^3\mu^4 + \cdots \end{aligned}$$

The periodic solution of equations (52) is

$$(53)\quad \begin{aligned} \rho = 1 - \tfrac{2}{3}\mu &+ (0.45139 + 0.39762 \cos\tau - 0.62500 \cos 2\tau)\mu^2 \\ &+ (-0.47647 + 1.04542 \cos\tau + 0.86090 \sin\tau + 0.46875 \cos 2\tau \\ &- 1.35318 \sin 2\tau - 0.16567 \cos 3\tau)\mu^3 + \cdots, \\ w = \tau &+ (-0.79524 \sin\tau + 0.85938 \sin 2\tau)\mu^2 \\ &+ (0.13882 - 3.25720 \sin\tau + 1.72180 \cos\tau + 0.27995 \sin 2\tau \\ &- 1.86062 \cos 2\tau + 0.19881 \sin 3\tau)\mu^3 + \cdots. \end{aligned}$$

Example 3. In this case M_2 revolves about M_1 in a circle of radius A_2 with an angular velocity N_2. Let N_2 be determined by the relation

$$N_2 - N = 2\nu.$$

A_2, R_2, and W_2 are determined by the relations

$$N_2^2 A_2^3 = k^2(M_1 + M_2),$$

$$R_2 = \sqrt{1 + A_2^2 + 2A_2 \cos 2\nu t} = \sqrt{1 + A_2^2 + 2A_2 \cos 2\tau},$$

$$\sin W_2 = \frac{A_2 \sin 2\tau}{R_2},$$

$$\cos W_2 = \frac{R_2^2 + 1 - A_2^2}{2R_2}.$$

Here k^2 is given by the relation

$$N^2 = k^2(M + M_1),$$

whence

$$k^2 = 0.09091.$$

It follows that

$$k^2 M_1 = 0.90909, \qquad k^2 M_2 = 0.09091,$$

$$A_2 = 1.01200\mu, \qquad a = 0.76910\mu.$$

The differential equations become

$$\begin{aligned}
\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} &= (0.50000 + 1.50000 \cos 2w)\rho\mu^2 \\
&+ (0.86523 \cos w + 1.44205 \cos 3w)\rho^2\mu^3 \\
&+ (0.33273 + 0.73940 \cos 2w + 1.29395 \cos 4w)\rho^3\mu^4 \\
&+ (0.27600 \sin 2\tau \sin 2w - 0.13800 \cos 2\tau - 0.41400 \cos 2\tau \cos 2w)\rho\mu^3 \\
&+ (0.10474 + 0.17456 \cos 4\tau + 0.31422 \cos 2w + 0.10471 \cos 4\tau \cos 2w \\
&\qquad - 0.55864 \sin 4\tau \sin 2w)\rho\mu^4 \\
&+ (0.07960 \sin 2\tau \sin w - 0.31840 \cos 2\tau \cos w - 0.53068 \cos 2\tau \cos 3w \\
&\qquad + 0.39801 \sin 2\tau \sin 3w)\rho^2\mu^4 + \cdots, \\
\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) &= -(1.50000 \sin 2w)\rho\mu^2 \\
&- (0.28841 \sin w + 1.44205 \sin 3w)\rho^2\mu^3 \\
&- (0.36970 \sin 2w + 1.29395 \sin 4w)\rho^3\mu^4 \\
&+ (0.27600 \sin 2\tau \cos 2w + 0.41400 \cos 2\tau \sin 2w)\rho\mu^3 \\
&- (0.31422 \sin 2w + 0.10471 \cos 4\tau \sin 2w + 0.55864 \sin 4\tau \cos 2w)\rho\mu^4 \\
&+ (0.02653 \sin 2\tau \cos w + 0.10613 \cos 2\tau \sin w + 0.39801 \sin 2\tau \cos 3w \\
&\qquad + 0.53068 \cos 2\tau \sin 3w)\rho^2\mu^4 + \cdots.
\end{aligned} \tag{54}$$

The periodic solution of equations (54) is

$$\begin{aligned}
\rho &= 1 - \tfrac{2}{3}\mu + (0.38889 + 0.72102 \cos \tau - \cos 2\tau)\mu^2 \\
&\quad + (-0.02616 + 2.09168 \cos \tau - 0.45400 \cos 2\tau - 0.30043 \cos 3\tau \\
&\qquad + 0.03450 \cos 4\tau)\mu^3 + \cdots, \\
w &= \tau + (-1.44204 \sin \tau + 1.37500 \sin 2\tau)\mu^2 \\
&\quad + (-6.29838 \sin \tau + 2.12066 \sin 2\tau + 0.36051 \sin 3\tau \\
&\qquad - 0.03881 \sin 4\tau)\mu^3 + \cdots.
\end{aligned} \tag{55}$$

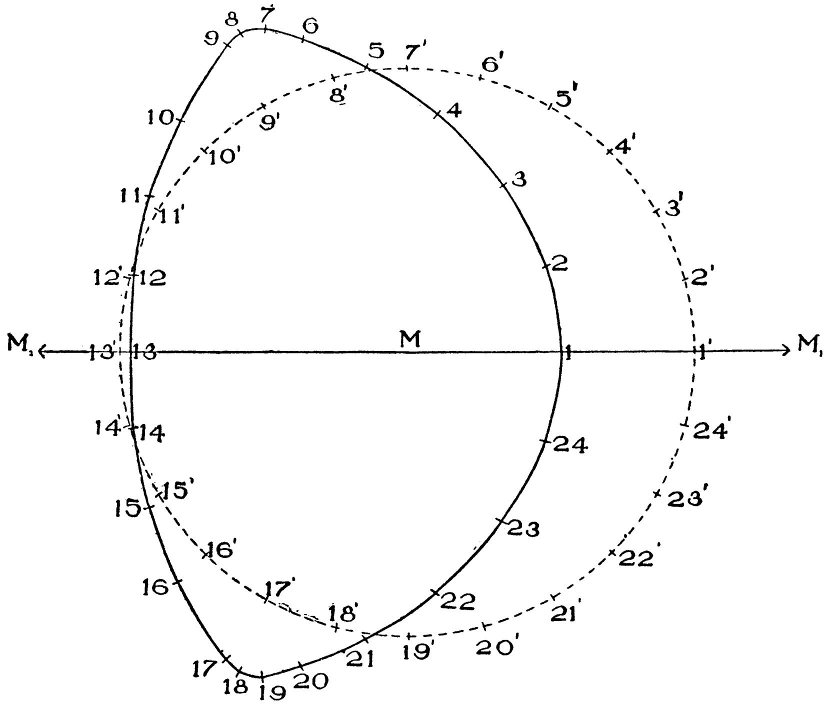

M
M,
M,
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
1' 2' 3' 4' 5' 6' 7' 8' 9' 10' 11' 12' 13' 14' 15' 16' 17' 18' 19' 20' 21' 22' 23' 24'

Fig. 2.

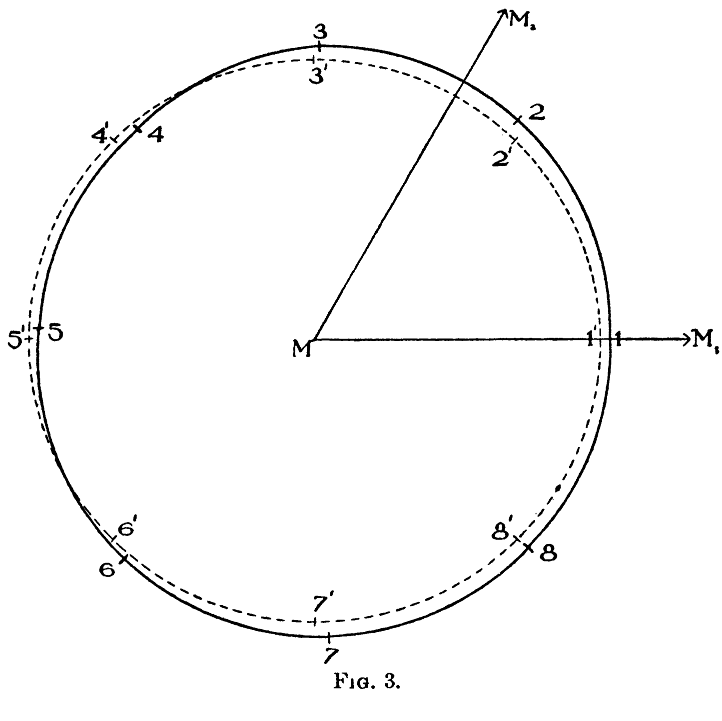

M
M,
M,
1 2 3 4 5 6 7 8
1' 2' 3' 4' 5' 6' 7' 8'

Fig. 3.

The orbits represented by the solutions (51), (53), and (55) are illustrated by figures 2, 3, and 4 respectively. The comparison circles in the figures are not the circular orbits which have been called the *undisturbed orbits.* The undisturbed orbits are referred to fixed axes while the drawings are made with reference to

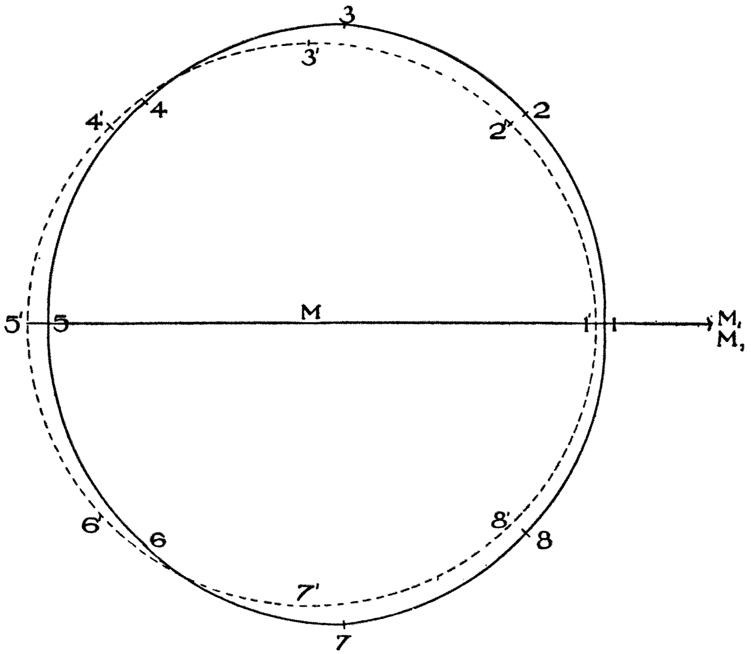


FIG. 4.

rotating axes. The comparison circles represent orbits in which the particle would make a complete revolution with respect to the rotating axes during the period. In figure 2 the points which are numbered 1, 2, $\cdots$ represent the positions of the particle in the periodic orbit at intervals in τ of $\pi/12$. The corresponding positions in the comparison circle are indicated by the numbers $1'$, $2'$, $\cdots$. In figures 3 and 4 the intervals are $\pi/4$.